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LETTER TO THE EDITOR

# Quantum groups constructed from the non-standard braid group representations in the Faddeev–Reshetikhin–Takhtajan approach

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**Abstract.** From the non-standard braid group representation for the spin-1 case of $sl_q(2)$, we find that the corresponding algebra constructed in the Faddeev–Reshetikhin–Takhtajan approach and allowed by the Yang–Baxter equation can be identified with the quantum group $Sl_\lambda(2)$ where $\lambda$ is a root of unity. Conversely, this suggests that quantum groups with $q$ being root of unity can be discussed using the non-standard braid group representation.

Much progress has been made in the study of the quantum group (QG) [1–7] which was developed for solving the quantum Yang–Baxter equation (QYBE) [8, 9] from the viewpoint of the quantum inverse scattering method [10–12]. The QYBE, which has its roots in the completely integrable systems, has emerged to be the common meeting ground of many beautiful branches of mathematics and physics [6, 7]. In the spectral-parameter-independent form, the QYBE reads

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{1}$$

Or, in terms of the $\check{R}$ matrix ($\check{R} = RP$, $P$ = permutation), $\check{R}$ satisfies the braid relation:

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}. \tag{2}$$

Faddeev, Reshetikhin and Takhtajan (FRT) have developed a powerful quantum $L$-operator formalism [4] which provides a direct connection between the quantum group (or Hopf algebra) and the $\check{R}$ matrix (satisfying the braid relation).

So far, two types of solution of (2) have been derived.

(A) Standard solutions. These correspond to the usual cases where the $\check{R}$ matrix can be constructed [2, 3] from the projectors (or equivalently from the Casimirs and the $q$-analogue Clebsch–Gordan coefficients). The standard solution possesses the usual classical limit $q \to 1$ [13]. For the standard solution, the FRT formalism gives the same QG as those discussed by Drinfeld [1] and Jimbo [2].

(B) Non-standard solutions. These are recognized as alternative solutions to the braid relation. Non-standard solutions for $sl_q(n)$ were first obtained by Lee *et al* [14]. They do not have the above-mentioned properties of the standard solutions. So far, very little is known regarding the QG associated with these non-standard solutions.

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In this letter, we study, in the FRT approach, the QG generated by the non-standard solution corresponding to the spin-1 case of $\mathrm{sl}_q(2)$.

Our result sheds some light on the role of such non-standard braid group representations (BGR), namely, they are associated with the QG at $q$ being a root of unity [15–17].

The basic relations in the FRT approach are the following three equations [4, 18]

$$(L^{\pm}\otimes L^{\pm})\check{R}=\check{R}(L^{\pm}\otimes L^{\pm}) \tag{3}$$

$$(L^{-}\otimes L^{+})\check{R}=\check{R}(L^{+}\otimes L^{-}) \tag{4}$$

where $L^{\pm}$ are $n\times n$ matrices whose elements are operators, and $\check{R}$ is a $n^2\times n^2$ matrix. If one inserts the standard solution $\check{R}$ (corresponding to a known QG), (4) would reproduce the known QG. Our main purpose here is to study the resulting algebraic structure when we insert the non-standard $\check{R}_{NS}$ for the spin-1 case of $\mathrm{sl}_q(2)$. The spin-$\frac{1}{2}$ case was recently discussed by Jing *et al* [19].

We consider the spin-1 case of $\mathrm{sl}_q(2)$. In appendix A, we give a summary of the two types of solution of the BGR for the spin-1 case as well as the spectral parameter-dependent $\check{R}(x)$. Here we just write down the results. The standard $R$ reads [20]:

$$\check{R}=\begin{bmatrix} 1 &&&&&&&& \\ & 0 & t &&&&&& \\ & t & 1-t^2 &&&&&& \\ &&& 0 & 0 & t^2 &&& \\ &&& 0 & t & t^{1/2}(1-t^2) &&& \\ &&& t^2 & t^{1/2}(1-t^2) & (1-t)(1-t^2) &&& \\ &&&&&& 0 & t & \\ &&&&&& t & 1-t^2 & \\ &&&&&&&& 1 \end{bmatrix}. \tag{5}$$

Here $t\equiv q^2$, $q$ is the parameter defined in $\mathrm{sl}_q(2)$. On the other hand, the non-standard $\check{R}_{NS}$ for this case reads [14, 21]:

$$\check{R}_{NS}=\begin{bmatrix} 1 &&&&&&&& \\ & 0 & t &&&&&& \\ & t & 1-t^2 &&&&&& \\ &&& 0 & 0 & t^2 &&& \\ &&& 0 & \omega t^2 & \mathrm{i}\omega t Z &&& \\ &&& t^2 & \mathrm{i}\omega t Z & Z^2 &&& \\ &&&&&& 0 & \omega^2 t^3 & \\ &&&&&& \omega^2 t^3 & -t^2(1-\omega t^2) & \\ &&&&&&&& \omega t^4 \end{bmatrix} \tag{6}$$

where

$$\omega\equiv \mathrm{e}^{2\pi\mathrm{i}/3} \qquad \text{is a root of unity} \tag{7}$$

$$Z^2\equiv(1-t^2)(1-\omega t^2). \tag{8}$$

We note in passing that for the spin-1 case, $\check{R}$ and $\check{R}_{NS}$ coincide only at a special value of $t$, namely at $t=1/\omega$. Such a limit

$$\check{R}=\lim_{t\to 1/\omega}\check{R}_{NS} \tag{9}$$

may serve as a consistency check on our final result.

We write $L^+$ and $L^-$ as $3\times3$ upper and lower triangular matrices respectively. This triangular nature of $L^\pm$ is closely related to the block triangular structure of the $\check{R}$ matrices in (5) or (6):

$$L^+=\begin{bmatrix} L^+_{11} & L^+_{12} & L^+_{13} \\ 0 & L^+_{22} & L^+_{23} \\ 0 & 0 & L^+_{33} \end{bmatrix} \tag{10}$$

$$L^-=\begin{bmatrix} L^-_{11} & 0 & 0 \\ L^-_{21} & L^-_{22} & 0 \\ L^-_{31} & L^-_{32} & L^-_{33} \end{bmatrix}. \tag{11}$$

By straightforward calculation, (3) and (4) produce a large set of constraints on the 12 operators $L^\pm_{ij}$. These relations are summarized in appendix B. Out of these relations, we find that the following basic algebraic structure is the key to all those relations stated in appendix B.

We find that $L^+$ and $L^-$ can be parametrized as follows.

$$L^+=\begin{bmatrix} K_1 & \tau X^+ & \mathrm{i}\omega^2\tau^{-1}ZK_1^{-1}(X^+)^2 \\ 0 & K_2 & -\mathrm{i}t^{-1}ZK_1^{-1}K_2X^+ \\ 0 & 0 & K_1^{-1}K_2^2 \end{bmatrix} \tag{12}$$

$$L^-=\begin{bmatrix} K_1^{-1} & 0 & 0 \\ -\tau X^- & K_2^{-1} & 0 \\ \mathrm{i}\omega^2\tau^{-1}ZK_1(X^-)^2 & \mathrm{i}t^{-1}ZK_1K_2^{-1}X^- & K_1K_2^{-2} \end{bmatrix} \tag{13}$$

where

$$\tau\equiv t-t^{-1}. \tag{14}$$

In $L^+$, it turns out that there are only two independent diagonal elements $K_1$, $K_2$ and *one* independent off-diagonal element $L^+_{12}$ (hereafter called $\tau X^+$). $L^+_{23}$ and $L^+_{13}$ are *dependent* variables. Likewise for $L^-$. The basic algebra reads:

$$(X^\pm)^3=0 \tag{15}$$

$$K_1X^\pm K_1^{-1}=t^{\pm1}X^\pm \tag{16}$$

$$K_2X^\pm K_2^{-1}=(\omega t)^{\pm1}X^\pm \tag{17}$$

$$[X^+,X^-]=\frac{1}{\tau}(K_1K_2^{-1}-K_2K_1^{-1}) \tag{18}$$

$$[K_1,K_2]=0. \tag{19}$$

The consistency requirement that $(X^\pm)^3$ commutes with everything can be verified.

It turns out that the algebra (15)-(19) can be simplified by a rescaling on the $K_i$. Let

$$\mathcal{K}^2\equiv K_1K_2^{-1} \tag{20}$$

and rescale $X^\pm$ by

$$X^\pm=\left(\frac{\lambda-\lambda^{-1}}{t-t^{-1}}\right)^{1/2}\tilde{X}^\pm \tag{21}$$

where

$$\lambda \equiv \omega^{-1/2} = e^{-\pi i/3} \tag{22}$$

and equation (15) retains the same form

$$(\tilde{X}^{\pm})^3 = 0. \tag{23}$$

Equation (18) now reads

$$[\tilde{X}^+, \tilde{X}^-] = \frac{\mathcal{K}^2 - \mathcal{K}^{-2}}{\lambda - \lambda^{-1}}. \tag{24}$$

The crucial feature is that (16) and (17) collapse into one equation:

$$\mathcal{K}^2 \tilde{X}^{\pm} \mathcal{K}^{-2} = K_1 K_2^{-1} \tilde{X}^{\pm} K_2 K_1^{-1} = \left(\frac{1}{\omega t}\right)^{\pm 1} K_1 \tilde{X}^{\pm} K_1^{-1}$$

$$= \left(\frac{1}{\omega}\right)^{\pm 1} \tilde{X}^{\pm}$$

or

$$\mathcal{K} \tilde{X}^{\pm} \mathcal{K}^{-1} = \lambda^{\pm 1} \tilde{X}^{\pm}. \tag{25}$$

We see that the algebra represented by (24) and (25) is nothing but the $\mathrm{sl}_q(2)$ except that the usual parameter $q$ (not being a root of unity) has now been replaced by $\lambda$ of (22) which is a root of unity. It is remarkable that the parameter $t$ has disappeared in the final algebra (25).

We have seen by an explicit example that from the non-standard $\check{R}_{NS}$ matrix for the spin-1 case, we recover in the FRT approach *not* the usual QG of $\mathrm{sl}_q(2)$, but $\mathrm{sl}_\lambda(2)$ where $\lambda$ is a root of unity. This suggests an interesting role played by such non-standard BGR.

We note that the condition such as (23) implies that the representation for $X^{\pm}$ satisfying $(X^{\pm})^{2j+1} = 0$ is just $(2j+1)$ dimensional (besides the more trivial ones). Such a condition is traceable to the appearance of the parameter $\omega$ satisfying $\omega^{2j+1} = 1$ in the non-standard BGR for spin $j$.

Our algebra (23)–(25) belongs to the so-called *modular* Hopf algebra recently discussed by Reshetikhin and Turaev [22]. In this sense, the present work gives an explicit example of such modular Hopf algebra.

It is instructive to compare our result for spin-1 case with the result of Lee [23] and with the recent result of [19] for the spin-$\frac{1}{2}$ case. In an entirely different approach, Lee [23] has discussed the $q$-deformation of $\mathrm{Sl}(2) \times Z_N$. With $\omega^N = 1$, his algebra is similar to our (16), (17). He had a different expression than our commutator (18), but this can be shown to be equivalent by using the parametrization of the Bazhanov–Stroganov ansatz [18]. On the other hand, his algebra does not contain $(X^{\pm})^{2j+1} = 0$ and he did not discuss the decoupling.

Our algebra (15)–(19) is similar to that of [19] upon identification of $(X^{\pm})^{2j+1} = 0$ and $\omega^{2j+1} = 1$. However, the authors of [19] did not discuss the decoupling leading to (25) as is done here.

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## Appendix A

For spin-1 of $sl_q(2)$, the BGR $\check{R}$ has the form

$$\check{R} = \text{block diag}[A_1, A_2, A_3, A_2', A_1'] \tag{A1}$$

where

$$A_1 = u \qquad A_1' = u' \qquad A_2 = \begin{bmatrix} 0 & p \\ p & w \end{bmatrix} \qquad A_2' = \begin{bmatrix} 0 & p' \\ p' & w' \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0 & 0 & p_0 \\ 0 & u_0 & r \\ p_0 & r & w_0 \end{bmatrix}. \tag{A2}$$

The parameters appearing in these matrices are to be determined by the braid relation. By an extended diagrammatic calculation, we obtain two types of solutions.

(A) Standard solution:

$$u = u' \qquad w = w' = u - u^{-1}p^2 \qquad p = p'$$
$$p_0 = u^{-1}p^2 \qquad w_0 = w(1 - u^{-1}u_0) \qquad r^2 = u_0^{-1}u^{-1}p^2w^2. \tag{A3}$$

Equations (A3) with $u = 1$, $u_0 = t$, $p = t$ give the solution shown in [20]. It belongs to the 'standard' family because it can be generated by the usual QG approach [2, 3].

(B) Non-standard solution:

$$w = u - u^{-1}p^2 \qquad w' = u' - u'^{-1}p'^2 \qquad u'p^2 = up'^2$$
$$p^{12} = u^9u'^3 \qquad u_0^2 = pp' = (u^2u'p^{-2})^2$$
$$w_0 = w(1 - u^{-1}u_0) \qquad p_0 = u^{-1}p^2$$
$$r^2 = u_0^{-1}u^{-1}p^2ww'. \tag{A4}$$

Equations (A4) admit a solution with

$$u = 1 \qquad u' = \omega t^4 \qquad \text{with } \omega^3 = 1, p = t \tag{A5}$$

that gives the non-standard solution used in (6) [14, 21].

To give the solution of the spectral parameter-dependent YBE, the Yang–Baxterization can be performed [24]. We can directly prove that the $\check{R}(x)$ given by

$$\check{R}(x) = x(x-1)\check{R}^{-1} + x(1 - t^{-2} + \omega^2t^{-4} - \omega^2t^{-2})I - (x-1)\omega^2t^{-4}\check{R} \tag{A6}$$

satisfies the YBE

$$\check{R}_{ij}^{ab}(x)\check{R}_{kf}^{jc}(xy)\check{R}_{de}^{ik}(y) = \check{R}_{ij}^{bc}(y)\check{R}_{dk}^{ai}(xy)\check{R}_{ef}^{kj}(x). \tag{A7}$$

In deriving (A6), the Yang–Baxterization prescription shown by [24] has been used. The $\check{R}^{-1}$ in (A6) has the same block structure as $\check{R}$, and can be obtained from $\check{R}$ by 'reflecting' each submatrix along the skew diagonal and letting $t \to t^{-1}$.

## Appendix B

We summarize the algebraic details from (3) and (4) with the insertion of $\check{R}_{NS}$ of (6) and $L^{\pm}$ from (10), (11).

The $9\times9$ matrix $\check{R}$, as written, is block diagonal. This corresponds to the following labelling of the composition of two spin-1 states: (1 1), (10, 01), (1 −1, 0 0, −1 1), (0 −1, −1 0), (−1 −1). On the other hand, in the usual direct product of $(1,0,-1)\otimes(1,0,-1)$, the labelling would come out as (1 1, 1 0, 1 −1), (0 1, 0 0, 0 −1), (−1 1, −1 0, −1 −1), which is not the same. So in order to conform to the basis in which the $\check{R}$ matrix is written, judicial permutations in $L^{\pm}\otimes L^{\pm}$ between the third and fourth, and between fifth and sixth, rows and columns are to be undertaken.

(1) Diagonal elements commute. (15 such non-trivial pairs)

$$[L^{\pm}_{ii}, L^{\pm}_{jj}]=0 \qquad i,j=1,2,3, \quad \text{all} \pm \text{combinations.} \tag{B1}$$

(2) Relations involving squares, and products vanishing:

$$(L^{+}_{13})^2=0 \qquad (L^{-}_{31})^2=0 \tag{B2}$$

$$(L^{+}_{12})^2=\frac{\mathrm{i}\omega Z}{t(1-\omega t^2)}L^{+}_{11}L^{+}_{13} \qquad (L^{-}_{21})^2=\frac{\mathrm{i}\omega Z}{t(1-\omega t^2)}L^{-}_{11}L^{-}_{31}. \tag{B3}$$

Use (B3) to express $L^{+}_{13}$ in terms of $(L^{+}_{12})^2$, the same for $L^{-}_{31}$:

$$(L^{+}_{23})^2=\frac{-\mathrm{i}Z}{t(1-t^2)}L^{+}_{33}L^{+}_{13} \qquad (L^{-}_{32})^2=\frac{-\mathrm{i}Z}{t(1-t^2)}L^{-}_{33}L^{-}_{31} \tag{B4}$$

$$(L^{+}_{22})^2=L^{+}_{11}L^{+}_{33} \qquad (L^{-}_{22})^2=L^{-}_{11}L^{-}_{33}. \tag{B5}$$

Use (B5) to express $L^{+}_{33}$ in terms of $L^{+}_{11}$ and $L^{+}_{22}$, the same for $L^{-}_{33}$:

$$L^{+}_{12}L^{+}_{13}=\omega^2t^3L^{+}_{13}L^{+}_{12}=0 \qquad L^{-}_{21}L^{-}_{32}=\omega^2t^3L^{-}_{32}L^{-}_{21}=0 \tag{B6}$$

$$L^{+}_{23}L^{+}_{13}=\omega t^3L^{+}_{13}L^{+}_{23}=0 \qquad L^{-}_{32}L^{-}_{31}=\omega t^3L^{-}_{31}L^{-}_{32}=0. \tag{B7}$$

Equations (B6), (B7) imply

$$(L^{+}_{12})^3=0 \qquad (L^{-}_{21})^3=0. \tag{B8}$$

Together with (B3), we see that (B2) is satisfied.

(3) Relations involving $L^{\pm}_{kk}$ on $L^{+}_{ij}$ and their conjugates:

$$\begin{aligned} L^{\pm}_{kk}L^{+}_{i\,i+1}(L^{\pm}_{kk})^{-1}&=(\omega^{k-1}t)^{\pm1}L^{+}_{i\,i+1} \\ L^{\pm}_{kk}L^{-}_{i+1\,i}(L^{\pm}_{kk})^{-1}&=(\omega^{k-1}t)^{\mp1}L^{-}_{i+1\,i} \end{aligned} \qquad k=1,2,3 \quad i=1,2 \tag{B9}$$

$$\begin{aligned} L^{\pm}_{kk}L^{+}_{13}(L^{\pm}_{kk})^{-1}&=(\omega^{k-1}t)^{\pm2}L^{+}_{13} \\ L^{\pm}_{kk}L^{-}_{31}(L^{\pm}_{kk})^{-1}&=(\omega^{k-1}t)^{\mp2}L^{-}_{31} \end{aligned} \qquad k=1,2,3. \tag{B10}$$

(4) Relations among $L^{+}L^{+}$; $L^{-}L^{-}$

$$L^{+}_{22}L^{+}_{12}=\frac{-\mathrm{i}Z}{(1-\omega t^2)}L^{+}_{11}L^{+}_{23} \qquad L^{-}_{22}L^{-}_{21}=\frac{-\mathrm{i}Z}{(1-\omega t^2)}L^{-}_{11}L^{-}_{32}. \tag{B11}$$

Use (B11) to express $L^{+}_{23}$ in terms of $L^{+}_{12}$, the same for $L^{-}_{32}$.

$$L^{+}_{22}L^{+}_{23}=\frac{\mathrm{i}Z}{(1-t^2)}L^{+}_{33}L^{+}_{12} \qquad L^{-}_{22}L^{-}_{32}=\frac{\mathrm{i}Z}{(1-t^2)}L^{-}_{33}L^{-}_{21} \tag{B12}$$

$$L^{+}_{22}L^{+}_{13}=-tL^{+}_{12}L^{+}_{23} \qquad L^{-}_{22}L^{-}_{31}=-tL^{-}_{21}L^{-}_{32}. \tag{B13}$$

(5) Relations among $L^+L^-$ combinations:

$$[L^+_{12}L^-_{21}] = -(t-t^{-1})(L^+_{11}L^-_{22} - L^+_{22}L^-_{11}). \qquad \text{(B14)}$$

This is the origin for (18).

$$L^+_{13}L^-_{21} - tL^-_{21}L^+_{13} = (t-t^{-1})L^+_{23}L^-_{11} + \mathrm{i}\omega Z L^-_{22}L^+_{12} \qquad \text{(B15)}$$

$$L^-_{31}L^+_{12} - tL^+_{12}L^-_{31} = (t-t^{-1})L^-_{32}L^+_{11} + \mathrm{i}\omega Z L^+_{22}L^-_{21} \qquad \text{(B16)}$$

$$L^+_{23}L^-_{21} - \omega^2 L^-_{21}L^+_{23} = \mathrm{i}t^{-1}Z(L^+_{22}L^-_{22} - L^+_{33}L^-_{11}) \qquad \text{(B17)}$$

$$L^-_{32}L^+_{12} - \omega^2 L^+_{12}L^-_{32} = \mathrm{i}t^{-1}Z(L^-_{22}L^+_{22} - L^-_{33}L^+_{11}) \qquad \text{(B18)}$$

$$L^+_{13}L^-_{32} - \omega^2 tL^-_{32}L^+_{13} = -\mathrm{i}\omega t^{-1}ZL^+_{23}L^-_{22} - (1-\omega t^2)L^-_{33}L^+_{12} \qquad \text{(B19)}$$

$$L^-_{31}L^+_{23} - \omega^2 tL^+_{23}L^-_{31} = -\mathrm{i}\omega t^{-1}ZL^-_{32}L^+_{22} - (1-\omega t^2)L^+_{33}L^-_{21} \qquad \text{(B20)}$$

$$[L^+_{13}, L^-_{31}] = -\mathrm{i}\omega t^{-1}Z(L^+_{23}L^-_{21} - L^-_{32}L^+_{12}) - t^{-2}Z^2(L^+_{33}L^-_{11} - L^-_{33}L^+_{11}) \qquad \text{(B21)}$$

$$[L^+_{23}, L^-_{32}] = \omega t^{-1}(1-\omega t^2)(L^+_{33}L^-_{22} - L^-_{33}L^+_{22}). \qquad \text{(B22)}$$

It is straightforward to verify that all these relations are satisfied with the basic algebra given by (12)–(19).

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